

Homological algebra solutions Week 9

1. Consider the first quadrant double complex C given by

$$\begin{array}{ccccccc}
 0 & \longleftarrow & C' & \longleftarrow & B' & \xleftarrow{\alpha} & A' & \longleftarrow & K' \\
 \downarrow & & \downarrow h & & \downarrow g & & \downarrow f & & \downarrow \\
 K & \longleftarrow & C & \xleftarrow{\beta} & B & \longleftarrow & A & \longleftarrow & 0
 \end{array},$$

where the rows are exact, and $K = \text{coker}(\beta)$, $K' = \ker(\alpha)$. We know that there exists a spectral sequence whose 0^{th} page is given by the above diagram (with vertical differentials), and who is converging to the homology of the total complex $H_*(T)$. Notice that since C has exact rows, the total complex of C is acyclic by the Acyclic Assembly Lemma. We now compute the pages $E_{\bullet,\bullet}^1, E_{\bullet,\bullet}^2$ of our spectral sequence. The first page is given by:

$$0 \longleftarrow \ker(h) \longleftarrow \ker(g) \longleftarrow \ker(f) \longleftarrow K'$$

$$K \longleftarrow \text{coker}(h) \longleftarrow \text{coker}(g) \longleftarrow \text{coker}(f) \longleftarrow 0$$

Taking homology, we get the second page :

$$\begin{array}{ccccccc}
 0 & & E_{1,1}^2 & & E_{1,2}^2 & & E_{1,3}^2 & & E_{1,4}^2 \\
 & \swarrow & & \swarrow & & \swarrow & & \swarrow & \\
 E_{0,0}^2 & & E_{1,0}^2 & & E_{2,0}^2 & & E_{3,0}^2 & & 0
 \end{array}$$

Now notice that except for $E_{1,1}^2$ and $E_{3,0}^2$, the differentials leaving and arriving at $E_{p,q}^2$ are 0, and this will remain the same for every page after. Since the sequence converges to 0, this means that $E_{p,q}^2 = E_{p,q}^\infty = 0$ for (p,q) not $(1,1)$ or $(3,0)$. But this implies that the sequences

$$\ker(h) \longleftarrow \ker(g) \longleftarrow \ker(f)$$

and

$$\text{coker}(h) \longleftarrow \text{coker}(g) \longleftarrow \text{coker}(f)$$

are exact.

We now investigate the map $E_{3,0}^2 \rightarrow E_{1,2}^2$. Its kernel (resp. cokernel) is given by the object $E_{3,0}^3$ (resp. $E_{1,2}^3$) of page 3. However, we can argue as before that these are the infinity terms, as no non zero differentials are ever going to reach or leave them in the later pages. This implies that $E_{3,0}^2 \rightarrow E_{1,2}^2$ is an isomorphism. Denote by $\sigma : \ker(g) \rightarrow \ker(h)$ and $\tau : \ker(f) \rightarrow \ker(g)$ the maps obtained before as differentials of the first page. We have that $\text{coker}(\sigma) \cong E_{3,0}^2 \cong E_{1,2}^2 \cong \ker(\tau)$, so we can glue our two sequences into the following exact sequence :

$$\begin{array}{ccccccc} \text{coker}(\sigma) & \xleftarrow{\eta} & \ker(h) & \xleftarrow{\sigma} & \ker(g) & \xleftarrow{\quad} & \ker(f) \\ & & & & \searrow \varphi & & \\ & & \text{coker}(h) & \xleftarrow{\quad} & \text{coker}(g) & \xleftarrow{\tau} & \text{coker}(f) \xleftarrow{\epsilon} \ker \tau \end{array}$$

Denote by δ the compositum $\epsilon \circ \varphi \circ \eta$. Note that $\ker(\delta) \cong \ker(\eta)$, as ϵ, φ are monic (for example using Freyd Mitchel's embedding theorem), so that $\ker(\delta) = \text{coker}(\sigma)$ by exactness. Similarly, we have that $\text{coker}(\delta) \cong \text{coker}(\epsilon)$ as η, φ are epic, which in turn implies $\text{coker}(\delta) \cong \ker(\tau)$. It follows that the sequence

$$\begin{array}{ccccc} \ker(h) & \xleftarrow{\quad} & \ker(g) & \xleftarrow{\quad} & \ker(f) \\ & & \searrow \delta & & \\ \text{coker}(h) & \xleftarrow{\quad} & \text{coker}(g) & \xleftarrow{\quad} & \text{coker}(f) \end{array}$$

is exact, which is exactly the Snake Lemma.

2. Let $\{E_{pq}^r\}_{p,q} \in \mathbb{Z}$ be a regular upper half-plane sequence with $E_{pq}^\infty \cong \mathbb{Z}/2\mathbb{Z}$ for any p, q with $q \geq 0$. Let H_* be the family of objects defined by $H_n = \mathbb{Z}$. Note that for $p \in \mathbb{Z}$ and $F_p = 2^p\mathbb{Z}$, we have a filtration

$$\dots \subseteq F_1 H_n \subseteq F_0 H_n = H_n,$$

which satisfies $F_p H_n / F_{p+1} H_n = 2^p \mathbb{Z} / 2^{p+1} \mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \cong E_{pq}^\infty$ for any p, q with $q \geq 0$. In other words, $\{E_{pq}^r\}$ weakly converges to H_* . Moreover, we see that this filtration is both Hausdorff and exhaustive

$$\bigcup_p F_p H_n = \bigcup_{p \geq 0} 2^p \mathbb{Z} = \mathbb{Z}, \quad \bigcap_p F_p H_n = \bigcap_{p \geq 0} 2^p \mathbb{Z} = 0,$$

so that $\{E_{pq}^r\}$ approaches H_* .

Now we set $H'_n = \mathbb{Z}_2$, where $\mathbb{Z}_2$ is by definition $\varprojlim \mathbb{Z}/2^p \mathbb{Z}$. Letting $F_p = 2^p \mathbb{Z}_2$ for $p \geq 0$, we find as before that $\{E_{pq}^r\}$ approaches H'_* . Now it is a classical computation in number theory that we have an isomorphism $\mathbb{Z}_2 / 2^p \mathbb{Z}_2 \cong \mathbb{Z}/2^p \mathbb{Z}$. Alternatively, it is a consequence of Proposition 10.15 in Atiyah-McDonald, Introduction to commutative algebra. It follows that our filtration is also complete :

$$\varprojlim H_n / F_p H_n = \varprojlim \mathbb{Z}_2 / 2^p \mathbb{Z}_2 \cong \varprojlim \mathbb{Z}/2^p \mathbb{Z} \cong \mathbb{Z}_2.$$

As $\{E_{pq}^r\}$ is regular, we deduce that $\{E_{pq}^r\}$ converges to H_* as wanted.

3. Let $n \geq 2$, and suppose the following is a Serre fibration

$$F \xrightarrow{\iota} E \xrightarrow{\pi} S^n .$$

We saw in class that there exists a first quadrant spectral sequence whose second page is given by $E_{p,q}^2 = H_p(S^n, H_q(F))$ and which converges to $H_{p+q}(E)$. We first compute these $E_{p,q}^2$ more explicitly. First, it is a classical result from topology that the p^{th} homology group of the n -sphere is given by :

$$H_p(S^n) = H_p(S^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & p = 0, n \\ 0 & \text{else.} \end{cases}$$

By the Universal Coefficient Theorem for Homology, we deduce that for $p \geq 1$:

$$H_p(S^n, H_q(F)) \cong H_p(S^n) \otimes H_q(F) \oplus \text{Tor}_1^{\mathbb{Z}}(H_{p-1}(S^n), H_q(F)) \cong \begin{cases} H_q(F) & p = 0, n \\ 0 & \text{else.} \end{cases} ,$$

where we have used that the Tor term is always 0, as $H_p(S^n)$ is either 0 or $\mathbb{Z}$ (which is a free $\mathbb{Z}$ -module, and thus in particular flat). On the other hand, when $p = 0$, we have

$$H_0(S^n, H_q(F)) = H_q(F)^{\pi_0(S^n)} = H_q(F).$$

We have just proven that the p^{th} homology group of S^n with coefficient in any abelian group G is given by G when $p = 0$ or n , and 0 otherwise. We deduce that the only non zero entries of the second page of the Serre spectral sequence are located in the 0^{th} and the n^{th} columns, where $E_{0,q}^2 = E_{n,q}^2 = H_q(F)$, for $q \geq 0$. Inspecting the shape of the differentials in the next pages, we see that the only differentials whose domain and codomain are both non zero are located on the n^{th} page :

$$d_{n,q}^n : E_{n,q}^n = H_q(F) \rightarrow H_{n+q-1}(F) = E_{0,n+q-1}^n$$

Since 0 and n are the only two non zero columns and the differentials on the later pages are always the zero map, we see that

$$\ker(d_{n,q}^n) = E_{n,q}^{n+1} = E_{n,q}^\infty, \quad \text{coker}(d_{n,q}^n) = E_{0,n+q-1}^{n+1} = E_{0,n+q-1}^\infty.$$

It follows that the sequence

$$0 \longrightarrow E_{n,q}^\infty \longrightarrow H_q(F) \longrightarrow H_{n+q-1}(F) \longrightarrow E_{0,n+q-1}^\infty \longrightarrow 0$$

is exact.

Now since our sequence converges to the homology groups of E , we have isomorphisms :

$$E_{p,q}^\infty \cong F_p H_{p+q}(E) / F_{p-1} H_{p+q}(E),$$

for some filtration

$$\dots \subseteq F_p H_{p+q}(E) \subseteq F_{p+1} H_{p+q}(E) \subseteq \dots \subseteq H_{p+q}(E)$$

of $H_{p+q}(E)$. We will now use the convergence of our spectral sequence to deduce some information on this filtration. Indeed, for $p \neq 0, n$, we know that

$$F_p H_{p+q}(E) / F_{p-1} H_{p+q}(E) \cong E_{p,q}^\infty = E_{p,q}^2 = 0.$$

Now by definition of convergence, the filtrations $(F_\bullet)$ are Hausdorff, so that $F_i H_{p+q}(E) \cong F_{-1} H_{p+q}(E)$ for all $i < 0$ implies that all these terms are zero. Since it is also exhaustive, we deduce that $F_i H_{p+q}(E) \cong H_{p+q}(E)$ for every $i \geq n$. Using additionally that all $F_i H_{p+q}(E)$ are isomorphic for $i = 0, \dots, n-1$, we deduce that the filtration is of the shape

$$0 = F_{-1} H_{p+q}(E) \subseteq F_1 H_{p+q}(E) \cong F_2 H_{p+q}(E) \dots \cong F_{n-1} H_{p+q}(E) \subseteq F_n H_{p+q}(E) \cong H_{p+q}(E).$$

In particular,

$$E_{n,q-n}^\infty \cong H_q(E)/F_0 H_q(E), \quad E_{0,q}^\infty \cong F_0 H_q(E)$$

so that

$$0 \longrightarrow E_{0,q}^\infty \longrightarrow H_q(E) \longrightarrow E_{n,q-n}^\infty \longrightarrow 0$$

is exact. Combining these two sequences with the “edge map” $\iota_* : H_q(F) = E_{0,q}^2 \twoheadrightarrow E_{0,q}^\infty \subseteq H_q(E)$ (see Addendum 1 to Theorem 5.3.2 in Weibel), we get the Wang sequence. The construction is summarized by the following diagram, where the Wang sequence is given by the middle row :

$$\begin{array}{ccccccccccccccc}
 & & & & & & 0 & & & & & & & & \\
 & & & & & & \uparrow & & & & & & & & \\
 \dots & \xrightarrow{d^n} & H_q(F) & \longrightarrow & E_{0,q}^\infty & \longrightarrow & 0 & \longrightarrow & E_{n,q-n}^\infty & \longrightarrow & H_{q-1}(F) & \xrightarrow{d^n} & H_{q-1}(F) & \longrightarrow & E_{0,q-1}^\infty & \longrightarrow & 0 \\
 & & \parallel & & & & & & \uparrow & & \parallel & & \parallel & & & & & \\
 & & H_q(F) & \xrightarrow{\quad \iota_* \quad} & H_q(E) & \longrightarrow & H_{q-1}(F) & \longrightarrow & H_{q-1}(F) & \xrightarrow{\quad \iota_* \quad} & \dots & & & & & & \\
 & & \parallel & & & & \uparrow & & \parallel & & & & \parallel & & & & & \\
 & & E_{0,q}^2 & \longrightarrow & E_{0,q}^\infty & & & & E_{0,q-1}^2 & \longrightarrow & \dots & & & & & & \\
 & & & & & & \uparrow & & & & & & & & & & \\
 & & & & & & 0 & & & & & & & & & &
 \end{array}$$

4. Let $n \geq 2$. As seen in class, the following is a Serre fibration :

$$\Omega S^n \longrightarrow PS^n \xrightarrow{\pi} S^n.$$

We then have the following long exact sequence (see paragraph before Theorem 5.3.2 in Weibel) :

$$\dots \longrightarrow \pi_{k+1}(S^n) \longrightarrow \pi_k(\Omega S^n) \longrightarrow \pi_k(PS^n) \longrightarrow \pi_k(S^n) \longrightarrow \dots$$

Since PS^n is contractible, we have that $\pi_k(PS^n) = 0$ for every $k > 0$. By exactness, this implies that $\pi_k(\Omega S^n) \cong \pi_{k+1}(S^n)$ for every $k > 0$. On the other hand, we have that

$$H_0(\Omega S^n) \cong \mathbb{Z}^{\pi_0(\Omega S^n)} \cong \mathbb{Z}^{\pi_1(S^n)} = \mathbb{Z}.$$

We will now recall a fundamental theorem of topology due to Hurewicz (Theorem 4.32 in J.Hatcher, Algebraic Topology) :

Theorem 0.1. *Let X be a path-connected space satisfying $\pi_k(X) = 0$ for every $k < n$. Then $H_i(X) = 0$ for every $0 < i < n$, and $\pi_n(X) \cong H_n(X)$.*

Now it is a classical result from topology that the homotopy groups of S^n are zero for k less than n . Combining Hurewicz theorem and the last exercise, we deduce that $\pi_n(S^n) \cong \mathbb{Z}$. This means that $\pi_k(\Omega S^n)$ is 0 for $0 < k < n - 1$, and $\pi_{n-1}(\Omega S^n) \cong \mathbb{Z}$. Applying Hurewicz theorem, we have obtained so far that

$$H_k(\Omega S^n) = \begin{cases} \mathbb{Z} & k = 0, n - 1 \\ 0 & 0 < k < n - 1 \end{cases}$$

For higher dimension, we use the Wang sequence. Once again, we have that $H_k(PS^n) = 0$ for every $k > 0$. This implies that for $k \geq n - 1$, we have

$$H_k(\Omega S^n) \cong H_{k-(n-1)}(\Omega S^n).$$

We deduce immediately from this that the k^{th} homology group of the loop space is isomorphic to the k_0^{th} one, where k_0 is the unique integer $0 \leq k_0 < n - 1$ satisfying $k_0 \equiv k \pmod{n - 1}$. By the first part, we deduce that

$$H_k(\Omega S^n) = \begin{cases} \mathbb{Z} & \text{if } (n - 1) \mid k \\ 0 & \text{else} \end{cases},$$

which concludes.